The Langevin Equation Expanded to 2nd order and comments on fitting the OmegaTau constants and Tensor Terms

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The Langevin equation for the drift velocity, $\vec{u}$, is given in Blum and Rolandi:

$$\vec{u} = \frac{\mu|\vec{E}|}{(1+\omega^2\tau^2)} \left[\hat{E} + \omega\tau \left(\hat{E} \times \hat{B}\right) + \omega^2\tau^2 \left(\hat{E} \cdot \hat{B}\right) \hat{B} \right]$$

where $\hat{E} = \frac{\vec{E}}{|\vec{E}|}$, $\hat{B} = \frac{\vec{B}}{|\vec{B}|}$ The components of the Langevin equation are:

$$u_x = \frac{\mu|\vec{E}|}{(1+\omega^2\tau^2)} \left[\hat{E}_x + \omega\tau \left(\hat{E}_y\hat{B}_z - \hat{E}_z\hat{B}_y\right) + \omega^2\tau^2 \left(\hat{E} \cdot \hat{B}\right) \hat{B}_x \right]$$

$$u_y = \frac{\mu|\vec{E}|}{(1+\omega^2\tau^2)} \left[\hat{E}_y + \omega\tau \left(\hat{E}_z\hat{B}_x - \hat{E}_x\hat{B}_z\right) + \omega^2\tau^2 \left(\hat{E} \cdot \hat{B}\right) \hat{B}_y \right]$$

$$u_z = \frac{\mu|\vec{E}|}{(1+\omega^2\tau^2)} \left[\hat{E}_z + \omega\tau \left(\hat{E}_x\hat{B}_y - \hat{E}_y\hat{B}_x\right) + \omega^2\tau^2 \left(\hat{E} \cdot \hat{B}\right) \hat{B}_z \right]$$

Let's assume that the E and B fields are nearly parallel and have principle components that lie along the Z axis with $\left|\vec{E}\right| \approx E_0$ and $\left|\vec{B}\right| \approx B_0$. B_0 could be the strength of the field at the center of the magnet and E_0 could be the drift field in the TPC. To be precise, let's assume small perturbations of the fields and define:

$$\vec{E} = \left(E_x, E_y, E_0 - \delta E_z\right), \qquad \vec{B} = \left(B_x, B_y, B_0 - \delta B_z\right)$$

so that $\hat{B}_x = \frac{B_x}{\sqrt{\left(B_x^2 + B_y^2 + \delta B_z^2 - 2B_0\delta B_z + B_0^2\right)}}$ but now define $\tilde{B}_x = \frac{B_x}{B_0}$ then

following the expansion to 3rd order, but keeping only 2nd order terms yields:

$$\hat{B}_x = \tilde{B}_x \left(1 + \delta\tilde{B}_z + \delta\tilde{B}_z^2 - \frac{1}{2}\tilde{B}_x^2 - \frac{1}{2}\tilde{B}_y^2\right) = \frac{B_x}{B_z} - \frac{1}{2}\tilde{B}_x^3 - \frac{1}{2}\tilde{B}_x\tilde{B}_y^2$$

and neglecting the 3rd order terms gives a very simple result:

$$\hat{B}_x = \frac{B_x}{B_z} \quad \text{and} \quad \hat{B}_y = \frac{B_y}{B_z} \quad \text{but also} \quad \hat{E}_x = \frac{E_x}{E_z} \quad \text{and} \quad \hat{E}_y = \frac{E_y}{E_z} \quad \text{with}$$

$$\hat{B}_z = \left(1 - \frac{1}{2}\hat{B}_x^2 - \frac{1}{2}\hat{B}_y^2\right) \quad \text{and} \quad \hat{E}_z = \left(1 - \frac{1}{2}\hat{E}_x^2 - \frac{1}{2}\hat{E}_y^2\right).$$

This means that

$$\hat{E} \cdot \hat{B} = \left(1 + \hat{E}_x\hat{B}_x + \hat{E}_y\hat{B}_y - \frac{1}{2}\hat{E}_x^2 - \frac{1}{2}\hat{E}_y^2 - \frac{1}{2}\hat{B}_x^2 - \frac{1}{2}\hat{B}_y^2\right)$$

Continuing the algebra and neglecting the 3rd order terms, we find that

$(\hat{E}\cdot\hat{B})\hat{B}_x = \hat{B}_x$ and $(\hat{E}\cdot\hat{B})\hat{B}_y = \hat{B}_y$ while

$$(\hat{E}\cdot\hat{B})\hat{B}_z = \left(1 + \hat{E}_x\hat{B}_x + \hat{E}_y\hat{B}_y - \frac{1}{2}\hat{E}_x^2 - \frac{1}{2}\hat{E}_y^2 - \hat{B}_x^2 - \hat{B}_y^2\right)$$

These equations are exact through 2nd order in all terms.

Returning to the Langevin equations, we find:

$$\frac{u_x}{u_z} = \frac{\left[\hat{E}_x + \omega\tau\left(\hat{E}_y\hat{B}_z - \hat{E}_z\hat{B}_y\right) + \omega^2\tau^2\,\hat{B}_x\right]}{\left[\hat{E}_z + \omega\tau\left(\hat{E}_x\hat{B}_y - \hat{E}_y\hat{B}_x\right) + \omega^2\tau^2\,(\hat{E}\cdot\hat{B})\hat{B}_z\right]}$$ or equivalently

$$\frac{u_x}{u_z} = \frac{\left[\frac{E_x}{E_z} + \omega\tau\left(\frac{E_y}{E_z} - \frac{B_y}{E_z}\right) + \omega^2\tau^2\,\frac{B_x}{B_z}\right]}{\left[\left(1 - O(2)\right) + \omega\tau\ O(2) + \omega^2\tau^2\left(1 + O(2)\right)\right]}$$

Note that the second order terms in the denominator will not survive the binomial expansion, because when they are brought to the numerator, they become 3rd order terms which we have agreed to neglect. Thus, the complete 2nd order equations are:

$$\frac{u_x}{u_z} = \frac{1}{(1+\omega^2\tau^2)}\frac{E_x}{E_z} + \frac{\omega\tau}{(1+\omega^2\tau^2)}\frac{E_y}{E_z} - \frac{\omega\tau}{(1+\omega^2\tau^2)}\frac{B_y}{E_z} + \frac{\omega^2\tau^2}{(1+\omega^2\tau^2)}\frac{B_x}{B_z}$$

$$\frac{u_y}{u_z} = \frac{1}{(1+\omega^2\tau^2)}\frac{E_y}{E_z} - \frac{\omega\tau}{(1+\omega^2\tau^2)}\frac{E_x}{E_z} + \frac{\omega\tau}{(1+\omega^2\tau^2)}\frac{B_x}{B_z} + \frac{\omega^2\tau^2}{(1+\omega^2\tau^2)}\frac{B_y}{E_z}$$

In a similar manner, we can evaluate the z component of the Langevin equation:

$$\frac{u_z}{u_0} = 1 - \frac{1}{2}\left(\hat{E}_x^2 + \hat{E}_y^2\right) + \frac{\omega\tau}{(1+\omega^2\tau^2)}\left(\hat{E}_x\hat{B}_y - \hat{E}_y\hat{B}_x\right) + \frac{\omega^2\tau^2}{(1+\omega^2\tau^2)}\left(\hat{E}_x\hat{B}_x + \hat{E}_y\hat{B}_y - \hat{B}_x^2 - \hat{B}_y^2\right)$$

where u_0 is the nominal drift velocity in the undistorted drift field, $|\vec{E}| = E_0$, and up to first order terms we find that the z component of the velocity is unperturbed.

Strictly speaking, these equations define the velocity of a drifting electron in nearly parallel E and B fields. In order to calculate the deviation of the electron away from a straight line path through these fields (i.e. the distortion in the transverse plane), these equations should be integrated over z.

For example, the distortions in the transverse plane are:

$$\delta_x = \int \frac{u_x}{u_z}\,dz, \quad \delta_y = \int \frac{u_y}{u_z}\,dz$$

and in the z direction the distortion is $\delta_z = \left(Z_{drift} - \int \frac{u_z}{u_0}\,dz\right)$.

Finally, lets simplify the notation, and define three universal constants

$c_0 = \frac{1}{(1+\omega^2\tau^2)}$, $c_1 = \frac{\omega\tau}{(1+\omega^2\tau^2)}$, and $c_2 = \frac{\omega^2\tau^2}{(1+\omega^2\tau^2)}$ in order to give:

<u>The complete 2nd order distortion equations in the transverse plane</u>

$$\delta_x = c_0 \int \frac{E_x}{E_z}\, dz + c_1 \int \frac{E_y}{E_z}\, dz - c_1 \int \frac{B_y}{B_z}\, dz + c_2 \int \frac{B_x}{B_z}\, dz$$

$$\delta_y = c_0 \int \frac{E_y}{E_z}\, dz - c_1 \int \frac{E_x}{E_z}\, dz + c_1 \int \frac{B_x}{B_z}\, dz + c_2 \int \frac{B_y}{B_z}\, dz$$

<u>and the complete 2nd order distortion equation in the z direction</u>

$$\delta_z = - \int \frac{E_x^2 + E_y^2}{2E_z^2}\, dz + c_1 \int \frac{E_x B_y - E_y B_x}{E_z B_z}\, dz + c_2 \int \left(\frac{E_x B_x + E_y B_y}{E_z B_z} - \frac{B_x^2 + B_y^2}{B_z^2} \right) dz$$

Note that it is important to use E_z and B_z in the denominator of each term, above, and not E_0 and B_0. If we were to use E_0 and B_0 we would be wrong by terms at the 2nd order for the corrections in the transverse plane.

It is nice to see that the distortion equations in the transverse plane are linear first order equations so that it is possible to calculate each source of distortion independently. Each perturbation can be handled as a separate calculation and added perturbatively to the sum of all the other distortions. Only the distortion in the z direction involves mixed fields, and this only happens at 2nd order.

The distortions in the transverse plane are separable, and by looking at the E field or B field distortions independently we find a formulation of these equations that looks like a rotation matrix. For example:

$$\begin{pmatrix} \delta_{xE} \\ \delta_{yE} \end{pmatrix} = \begin{pmatrix} c_0 & c_1 \\ -c_1 & c_0 \end{pmatrix} \begin{pmatrix} \int \frac{E_x}{E_z}\, dz \\ \int \frac{E_y}{E_z}\, dz \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \delta_{xB} \\ \delta_{yB} \end{pmatrix} = \begin{pmatrix} c_2 & -c_1 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} \int \frac{B_x}{B_z}\, dz \\ \int \frac{B_y}{B_z}\, dz \end{pmatrix}$$

The matrix equations, above, also work in cylindrical coordinates if you map x⇒r, y⇒ϕ, and δx⇒δr, δy⇒rδϕ.

A trivial example of the use of these matrix equations is to calculate the distortion due to a misalignment between the E and B fields. Suppose that the fields are not parallel but are twisted by a small angle θ and assume that the E field lies precisely on the Z axis as is the case for the TPC. In this example, we find:

$$\begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} c_2 & -c_1 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} \theta_x \ Z_{drift} \\ \theta_y \ Z_{drift} \end{pmatrix}$$

How good are the 1^{st} and 2^{nd} order equations? Typically, they are very good. For example, consider the magnetic field inside the ALICE TPC. The B field fluctuations are fairly large but do not exceed ±100 gauss in a 5000 gauss field. To be precise, the largest radial components of the field are ±100 gauss at the outer radius of the TPC. The ϕ components of the field are ±50 gauss, and the z component of the field runs from a low of 4900 gauss to a high of 5050 gauss. So the B field variations can be as large as 2% giving rise to 1 cm scale distortions. This means that a 1 cm distortion can be calculated, in first order, to a precision of 200 μm and the 2^{nd} order terms should be good to 4 μm. In comparison, it rarely happens that E field fluctuations are so large in a TPC. Even an electrical short between two strips in the field cage, which is about the worst thing that can happen, will introduce ~1% field gradients. Thus, the E field and B field distortions calculated to 2^{nd} order are good enough for even the most precise work.

The only remaining question is whether the Langevin equation is a complete description of the microscopic physics, and whether the constants c_i are correct as written above. The answer is that the Langevin equation is, in fact, the governing equation if you update the values of the constants c_i … I won't go into the details here, but the arguments are given in Blum, Riegler and Rolandi sections 2.3.3 and 2.4.4. They show that the Langevin equation is valid but the constants of the motion need to be updated because in the full microscopic theory, the drift velocity is a tensor and the drift velocities in the transverse plane are slightly different than the drift velocity along the z axis. To make a long story short, they show that the constants of motion should be written in the following way:

$$c_0' = \frac{1}{(1+T_2^2\omega^2\tau^2)}\ ,\ c_1' = \frac{T_1\omega\tau}{(1+T_1^2\omega^2\tau^2)},\quad \text{and}\quad c_2' = \frac{T_2^2\omega^2\tau^2}{(1+T_2^2\omega^2\tau^2)}$$

where T_1 and T_2 are the modifiers to the distortion equations due to the tensor terms in the drift velocity. Rolandi et al. have measured these tensor terms in P9 gas with a small test chamber and found $T_1 = 1.34$ and $T_2 = 1.11$ while STAR has measured them to be $T_1 = 1.36$ and $T_2 = 1.11$ in P10 gas. I am not aware of anyone measuring these tensor terms for Neon based gases and so they will have to be derived from the data. The best way to do this is to work with a known distortion and to try to remove it from the data. It is possible to do this with electric field distortions, alone, because there are only two unknown tensor terms and they are both used in the E field distortion matrices. A gating grid voltage scan might be a good way to do this. If a known B field distortion can be found, the results of the E field fit can be tested for consistency because the same tensor terms appear in the B field distortion matrices.

Conclusions:

Distortions in the transverse plane are accurately modeled to 2nd order using the following equations:

$$\begin{pmatrix} \delta_{xE} \\ \delta_{yE} \end{pmatrix} = \begin{pmatrix} c_0 & c_1 \\ -c_1 & c_0 \end{pmatrix} \begin{pmatrix} \int \frac{E_x}{E_z}\, dz \\ \int \frac{E_y}{E_z}\, dz \end{pmatrix}$$

$$\begin{pmatrix} \delta_{xB} \\ \delta_{yB} \end{pmatrix} = \begin{pmatrix} c_2 & -c_1 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} \int \frac{B_x}{B_z}\, dz \\ \int \frac{B_y}{B_z}\, dz \end{pmatrix}$$

The tensor terms can be measured by fitting the constants of motion, c_i, to real data. Thereafter, all other distortions can be calculated by using the following form for the constants of motion (dropping the prime notation):

$$c_0 = \frac{1}{(1+T_2^2\omega^2\tau^2)}\,, \quad c_1 = \frac{T_1\omega\tau}{(1+T_1^2\omega^2\tau^2)}, \quad \text{and} \quad c_2 = \frac{T_2^2\omega^2\tau^2}{(1+T_2^2\omega^2\tau^2)}$$

with

$$\omega\tau = -10.0 * BField[kGauss] * \frac{Drift\ Velocity\ [cm/\mu\text{sec}]}{Electric\ Drift\ Field\ Strength\ [V/cm]}$$

Note that $\omega\tau$ is a signed quantity and the sign of the B field is important in order to obtain the correct sign for the constants of the motion when the B field polarity is reversed.

In practical application, the integrals of the fields can be pre-computed and put into a C++ class. This allows for fast computation of the distortions and yet the constants of motion are completely independent of the integrals and can be updated run by run as the drift velocity changes.